Generalized Riemann - Liouville fractional derivatives for multifractal sets

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The Riemann-Liouville fractional integrals and derivatives are generalized for cases when fractional exponent d are functions of space and times coordinates (i.e. $d = d(r(t), t)$).

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I. INTRODUCTION

Fractional derivatives and integrals (left-sided and right-sided) Riemann - Liouville (see [1]- [3]) from functions $f(t)$ (defined on a class of generalized functions) are

$$D^{d}_{+,t}f(t) = \frac{1}{\Gamma(n-d)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}\frac{f(t')dt'}{(t-t')^{d-n+1}} \tag{1}$$

$$D^{d}_{-,t}f(t) = \frac{(-1)^{n}}{\Gamma(n-d)}\left(\frac{d}{dt}\right)^{n}\int_{t}^{b}\frac{f(t')dt'}{(t'-t)^{d-n+1}} \tag{2}$$

where $\Gamma(x)$ is Euler's gamma function, and a and b are some constants from $[0,\infty)$. In these definitions, as usually, $n = \{d\} + 1$, where $\{d\}$ is the integer part of d if $d \geq 0$ (i.e. $n-1 \leq d < n$) and $n = 0$ for $d < 0$. Fractional derivatives and the integrals (1)-(2) allow to use, instead of usual derivatives and integrals, the integral functionals defined on a wide class of generalized functions. It is very useful for the solution of a series of problems describing stochastic and chaos processes, abnormal diffusion, quantum theories of a field etc. [4]- [10]. It is possible to consider appearance of integral in (1)-(2), from the physical point of view, as the result of taking into account influence of the contributions from some physical processes (characterized by the kernel $(t-t')^{-d+n-1}\Gamma^{-1}(n-d)$) in earlier (left-side derivative) or later (right-hand derivative) times, on function $f(t)$ that is, as the partial taking into account the system memory about past or future times. The value of fractional exponent d characterizes the degree of the memory. Let's consider multifractal set (without self-similarity) S_t consisting from infinite number of subsets $s_i(t_i)$, also being multifractal. Each subset $s_i(t_i)$ is compared with fractional value (or number of values), describing its fractal (fractional) dimension (box dimension, Hausdorff [11] or Renie [12] dimension etc. - see, for example, [13]), depending from the numbers of a subset $s_i(t)$. Let the carrier of measure of multifractal set S_t be the set R^n. For exposition of changes of a continuous function $f(t)$ defined on subsets $s_i(t_i)$ of set S_t, it is impossible to use ordinary derivatives or Riemann - Liouville fractional derivatives (1), as the fractional dimension of sets d on which $f(t)$ is defined depends on t_i, that is on the choice of the subset $s_i(t_{i)}$. There is a problem: how can the definition (1)-(2) be changed to feature small (or major) changes of function $f(t)$ defined on sets $s_i(t_{i)}$? The purpose of this paper is to present the generalization of the Riemann - Liouville fractional derivatives (1)-(2) in order to ajust them for functions defined on multifractal sets with fractal dimension (fractional dimension) depending on the coordinates and time.

II. GENERALIZED FRACTIONAL DERIVATIVES AND INTEGRALS

We shall treat subsets $s_i(t_{i)}$ as the "points" t_i (with a continuous distribution for different multifractal subsets $s_i(t_{i)})$ of multifractal set S_t). Assume that the function $d(t_i) = d(t)$, describing their fractional dimension (in some cases coinciding with local fractal dimension) as function t is continuous. For the elementary generalization (1)-(2) is used physical reasons and variable t is interpreted as a time. For continuous functions $f(t)$ (generalized functions defined on the class of finitary functions (see [3])), the Riemann - Liouville fractional derivatives also are continuous. So for infinitesimal intervals of time and the functionals (1)-(2) will vary on infinitesimal quantity. For continuous function $d(t)$ the changes thus also will be infinitesimal. It allows, as the elementary generalization (1) suitable for describing of changes $f(t)$ defined on multifractal subsets $s(t)$, as well as in (1)-(2), to summate influence of a kernel of integral $(t-t')^{-d(t)-n+1}\Gamma^{-1}(n-d(t))$, depending on $d(t)$, in all points of integration and, instead of (1)-(2), introduce the following definitions (generalized fractional derivatives and integrals (GFD)), taking into account also the $d(t)$ dependence on time and vector parameter $\mathbf{r}(t)$ (i.e. $d_t \equiv d_t(\mathbf{r}(t), t)$)

$$D^{d_t}_{+,t}f(t) = \left(\frac{d}{dt}\right)^{n}\int\limits_{a}^{t}dt'\frac{f(t')}{\Gamma(n-d_t(t'))(t-t')^{d_t(t')-n+1}} \tag{3}$$

$$D^{d_t}_{-,t}f(t) = (-1)^{n} \times$$

$$\times \left(\frac{d}{dt}\right)^n \int_t^b dt' \frac{f(t')}{\Gamma(n - d_t(t'))(t' - t)^{d_t(t') - n + 1}} \qquad (4)$$

In (3)-(4), as well as in (1)-(2), a and b stationary values defined on an infinite axis (from $-\infty$ to ∞), $a < b$, $n - 1 \le d_t < n$, $n = \{d_t\} + 1$, $\{d_t\}$- the whole part of $d_t \ge 0$, $n = 0$ for $d_t < 0$. The sole difference (3)-(4) from (1)-(2) is: $d_t = d_t(\mathbf{r}(t), t)$- fractional dimension (further will be used for it terms " fractal dimension " (FD) or " the local fractal dimension (LFD) ") is the function of time and coordinates, instead of stationary values in (1)-(2).

Similar to (3)-(4), it is possible to define the GFD, (coinciding for integer values of fractional dimension $d_{\mathbf{r}}(\mathbf{r}, t)$ with derivatives with respect to vector variable $\mathbf{r}$) $D_{+,\mathbf{r}}^{d_{bfr}} f(\mathbf{r}, t)$ respect to vector $\mathbf{r}(t)$ variables (spatial coordinates). We pay attention, that definitions (3)-(4)are a special case of Hadamard derivatives [14].

III. FRACTIONAL DERIVATIVES FOR $D(\mathbf{R}(T), T) \to 1$

For FD which have very small differences from of integer values it is possible approximate to change the GFD by the usual derivatives and integrals. For an establishment of connection of GFD with orderly derivatives we shall see (3), for example, for a case $d(\mathbf{r}(t), t) = 1 + \varepsilon(\mathbf{r}(t), t)$,$\varepsilon \ll 1$, $d < 1$, (if utilize the theorem of the mean value of integral) as

$$D_{+,t}^{1-\varepsilon} f(t) = \frac{\partial}{\partial t} \int_0^t \frac{f(t-\tau)d\tau}{\Gamma(\varepsilon(t-\tau))(\tau \pm i\xi)^{1-\varepsilon(t-\tau)}} =$$
$$= \frac{\partial}{\partial t}[\tilde{f}(t - \tau_{cp}(t)) \int_0^t \frac{d\tau}{(\tau)^{1-\varepsilon(t-\tau)}}] \qquad (5)$$

where $\tilde{f} = \Gamma^{-1} f$ and t_{med}- some value of τ. As $\varepsilon \to 0$ it is possible to estimate values of integral in (5) for minimum and maximal values of ε ($\varepsilon > 0$)

$$\int_0^t \frac{d\tau}{\tau^{1-\varepsilon_{\min}(t-\tau)}} = \frac{t^{\varepsilon_{\min}}}{\varepsilon_{\min}}, \quad \int_0^t \frac{d\tau}{\tau^{1-\varepsilon_{\max}(t-\tau)}} = \frac{t^{\varepsilon_{\max}}}{\varepsilon_{\max}} \quad (6)$$

For selection from integrals (6) the trms which are independent from ε (because of $\tilde{f} \sim \varepsilon$) we use decomposition $t = 1 + \ln t + ...$ We obtain

$$D_{+,t}^{1-\varepsilon} f(t) \approx \frac{\partial f(t)}{\partial t} + \frac{\partial \tilde{f}(t - \tau_{cp}(t)}{\partial t} \ln t + \frac{\tilde{f}(t - \tau_{cp}(t)}{t} \qquad (7)$$

For major times $t = t_0 + (t - t_0)$,$t - t_0 \ll t_0$ the approximate representation GFD (7) through usual derivatives will accept a view (if neglect by additions of order $\tilde{f}/t_0$,$(t - t_0)/t_0$, to use the designation $\ln t_0 = \alpha$ and if it is accounted, that $\tau_{cp} \ll t$ because of the basic contribution to integral (5) is stipulated by small τ)

$$D_{+,t}^{1-\varepsilon} f(t) \approx \frac{\partial f(t)}{\partial t} + \frac{\partial \alpha \tilde{f}(t)}{\partial t} \qquad (8)$$

In (8) α play a role of a stationary value of parameter of regularization and if change ε (in $\tilde{f} = \varepsilon f$) on quantity $\varepsilon \to \varepsilon\alpha^{-1}$, GFD (8) is not depends practically on this parameter.

We shall give below, another method of deduction the relation (8) using an expansion $\tau^{1-\varepsilon}$ in a power series of ϵ under sign of integral and again for $d(r(t), t)$ with poorly difference from the whole value (but $d > 1$). Let's fractional dimension $d(\mathbf{r}(t), t)$ is equal unity with small value ε ($d(\mathbf{r}(t), t) = 1 + \varepsilon(\mathbf{r}(t), t)$,$\varepsilon \ll 1$) and expand FD in (3) in a power series on ε by a rule $(t - \tau)^{-\varepsilon} = 1\varepsilon \ln(t - \tau) + ...$ Restricted expansion FD by the first two members of a series, we obtain for left-side fractional derivative (for $a = 0$)

$$D_{+,t}^{1+\varepsilon} f(t) = \frac{\partial^2}{\partial t^2} \int_0^t \frac{f(\tau)}{\Gamma(1-\varepsilon)(t-\tau)^{\varepsilon(r(\tau),\tau)}} d\tau \approx$$
$$\approx \frac{\partial}{\partial t}(\frac{1}{\Gamma(1-\varepsilon)} f(t)) - \frac{\partial^2}{\partial t^2} \int_0^t \frac{\varepsilon f(\tau) d\tau}{\Gamma(1-\varepsilon)[(t-\tau) \pm i\varsigma]},$$
$$\varsigma \to 0 \qquad (9)$$

Integral in (9) is considered as a generalized function with determined regularizasion and after regularization of integral the parameter of regularization α is picked by a requirement of the best coincidence of approximate and exact results of integral calculation the members of a first order at ε (it is necessary to take its real part, the parameter ς is necessary to put zero after calculations). After an integration by parts and also using of a relation $1/x = P(1/x)$ or another regularization's we shall receive (if take into account that integrals (1)-(2) are real values and define the fractal addendum to derivatives as a coefficients at the imaginary parts of integrals)

$$D_{+,t}^{1+\varepsilon} f(t) = \frac{\partial}{\partial t}[\frac{1}{\Gamma(1-\varepsilon)} f(t)] \pm \frac{\partial}{\partial t}[\alpha \frac{\varepsilon(t) f(t)}{\Gamma(1-\varepsilon(t))}] \quad (10)$$

were α defined by selection of regularization .The selection of a sign in (6) is defined by a selection of the regularization. From (10) the opportunity follows (at the small fractional additives to FD of time) to use for describing of changes of functions defined on multifractal sets of time by means of using the renormalized ordinary derivatives. At the same time, the dependence FD of the time from coordinates and time is concerned. Let's consider fractional dimension d for case when d smaller of unity ($d = 1+\varepsilon$,$d < 1$). For this case fractional derivative (see (3) for $n = 1$) looks like

$$D^{1-\varepsilon}_{+,t}f(t)=\frac{\partial}{\partial t}\int\limits_0^t \frac{f(\tau)d\tau}{\Gamma(\varepsilon(\tau))(t-\tau\pm i\xi)^{1-\varepsilon(\tau)}} \qquad (11)$$

Taking into account, that for (11) selection ε, by virtue of definition (3)-(4)), is prohibited, for including in (11) value $D^{1-\varepsilon}_{+,t}f(t)$ at $\varepsilon=0$ before a right member in (11) (applicable only for $\varepsilon>0$) it is necessary to take into account a addendum from (9) with $\varepsilon=0$, i.e. $\frac{\partial f(t)}{\partial t}$. We receive (if use a rule of a regularization that was had used before for $d>1$ i.e. $Reg(1/x)\rightarrow\alpha\delta(x)$ and relation $\Gamma(1+\varepsilon)=\varepsilon\Gamma(\varepsilon)$)

$$\frac{\partial^{1-\varepsilon}}{\partial t^{1-\varepsilon}}f(t)\approx\frac{\partial}{\partial t}f(t)\mp\frac{\partial}{\partial t}[\alpha\frac{\varepsilon(r(t),t)f(t)}{\Gamma(1+\varepsilon(r(t),t))}f(t)] \qquad (12)$$

The approximate representation GFD by ordinary derivatives (relations (8),(10),(12)) if use different methods are very similar, so any of them may be used in follow calculations. The above mentioned approximate connections of generalized fractional derivatives (3)-(4) defined on the multifractal sets with fractional dimension $d(\mathbf{r}(t),t)$ (if $d(\mathbf{r}(t),t)$ poorly distinguished from unity) with ordinary derivatives may be ex-tend for the cases with arbitrary n:$d(\mathbf{r}(t),t)=n+\varepsilon(\mathbf{r}(t),t)$, $|\varepsilon|\ll1$. Above mentioned reasoning make possible to show, for a cases $\varepsilon\sim1$ (but not close to integer values), that the representations of generalized fractional derivative by means of derivatives of integer order will contain integer derivatives of arbitrary high orders. Let's consider a symmetrical generalized fractional derivatives $D^d_{-,t}f(t)$ and $D^d_{+,t}f(t)$:

$$D^d_tf(t)=0.5(D^d_{+,t}+D^d_{-,t})f(t) \qquad (13)$$

The symmetry of GFD allows to take into account the influence on event that happens in the given instant featured by function $f(t)$ both past, and future (by fractional integration and differentiation on time). For fractional integration and differentiation at coordinates the symmetrical GFD takes into account influence the event with given coordinate of all points of space

$$D^d_{\mathbf{r}}f(t)=0.5(D^d_{+,\mathbf{r}}+D^d_{-,\mathbf{r}})f(t)$$

At small difference of dimensions of time (or space) from unity $D^{1+\varepsilon}_{+,t}\approx D^{1+\varepsilon}_{-,t}$ and so on.

IV. CONNECTION WITH COVARIANT DERIVATIVES

Let define (10) as

$$D^{1+\varepsilon}_{+,t}f(t)\approx A\frac{\partial}{\partial t}f-Bf \qquad (14)$$

where

$$A(\mathbf{r}(t),t)=\Gamma(1-\varepsilon)^{-1}+a\varepsilon \qquad (15)$$

$$B(\mathbf{r}(t),t)=\Gamma(1-\varepsilon)^{-2}(1+a\varepsilon)\frac{\partial\Gamma}{\partial t}-a\frac{\partial\varepsilon}{\partial t},\quad (a=\pm1) \qquad (16)$$

The relation (16) reminds the covariant derivatives, frequently meeting in physics. It is possible to show, that at a various selection of a mathematical nature of function f (vector, tensor etc.) and relevant selection of function ε, GFD (10) really coincides with covariant derivatives (see [15], [16]).

V. EQUATIONS WITH GENERALIZED FRACTIONAL DERIVATIVES

The equations with GFD are possible to connect with natural sciences (in particular, physics) when the fractal dimensions d_t and $d_{\mathbf{r}}$ are connected with describing multifractal structure of a surfaces of solid bodies, structure of chaos, structure of time and space (see, for example, [5], [7], [8], [11], [16]. In some cases GFD are related to equations with FD that depends from functions (or functionals) the same to which GFD was applied. It gives in the interesting nonlinear fractional integral-differential equations with GFD

$$F(D^{d_t(f(t))}_{+,t})f(t)=0 \qquad (17)$$

where F- function or functional from GFD. A new class of the equations in fractional integral-differential functionals represent the equations such as (13). Their examination, apparently, is an interesting problem and represents a new approach to describe problems of chaos.

VI. CONCLUSION

The generalized Riemann-Liouville fractional derivatives defined in the paper allow to describe dynamics and changes of functions defined on multifractal sets, in which every element of sets is characterized by its own fractional dimension (depending on coordinates and time). At small differences of fractional dimensions from topological dimensions, generalized fractional derivatives are represented through expressions similar to covariant derivatives used in physics.

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